

Two Dimensional Mirror Symmetry from M-theory

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We construct two dimensional gauge theories with $N = (4, 4)$ supersymmetry from branes of type IIA string theory. Quantum effects in the two dimensional gauge theory are analyzed by embedding the IIA brane construction into M-theory. We find that the Coulomb branch of one theory and the Higgs branch of a mirror theory become equivalent at strong coupling. A relationship to the decoupling limit of the type IIA and IIB 5-branes in Matrix theory is shown. T-duality between the ALE metric and the wormhole metric of Callan, Harvey, and Strominger is discussed from a brane perspective and some puzzles regarding string duality resolved. We comment on the existence of a quantum Higgs branch in two dimensional theories. Branes prove to be useful tools in analyzing singular conformal field theories.

1. Introduction

There has been much progress made recently in understanding field theory from brane constructions. In [1], three dimensional gauge theories with 8 real supercharges were studied from type IIB string theory using D-branes and Neveu-Schwarz 5-branes. The field theory results of [2,3,4,5] were obtained from string theory and generalized. Specifically, the mirror symmetry of three dimensional gauge theories which relates hypermultiplets and vector multiplets of two different theories was seen as a result of the S-duality of type IIB. The brane construction of three dimensional gauge theories was generalized to constructions of five dimensional [6] and six dimensional [7] gauge theories with 8 real supercharges. Non-trivial fixed points of these gauge theories were derived from string theory and associated with tensionless branes. In four dimensions, brane constructions in type IIA string theory were found to be very useful in naturally providing for generalizations of Seiberg-Witten type solutions [8]. More recently, a IIB brane construction was used to study 0+1 dimensional theories in relation to Matrix theory [9].

In this paper, we examine brane constructions of two dimensional gauge theories with 8 real supercharges. N_c D2-branes are suspended between two NS 5-branes with N_f D4 branes intersecting the D2-branes in two dimensions. The theory on the intersection is then an $U(N_c)$ gauge theories with N_f flavors. The coupling constant of the theory on the D2-brane is inversely related to the distance between the NS 5-branes as well as directly related to the IIA string coupling constant, g_s . Since the D2-brane tension is much smaller than the D4 brane tension, the gauge symmetry on the D4 branes appears as a global symmetry of the 1+1 dimensional theory on the intersection. We extract quantum information about these two dimensional gauge theories by making the IIA string coupling constant large; that is, we embed the ten dimensional IIA brane construction into eleven dimensional M-theory. We find that the metric on the Coulomb branch of the $U(N_c)$ gauge theory receives quantum corrections, deforming it to the so called wormhole metric of [10]. This result was found by field theory techniques in [11]. We also find that there is another theory, $\prod_i U(n_i)$ with bi-fundamental matter fields, having an ALE-type metric with $\theta = 0$ which flows to the same wormhole metric in the IR. This duality is reminiscent of Seiberg's "non-Abelian Coulomb phase" in four dimensions [12] and mirror symmetry in two dimensions [13]. In this respect, this duality is different from the mirror symmetry found in three dimensional gauge theories which relates strong to weak coupling.

In section two, we present the type IIA brane construction of $N = (4, 4)$ two dimensional gauge theories. In section three, we study quantum effects on the world volume

theory of the D2-brane by considering the bending of the branes in IIA and by making the eleventh dimension very large. The torsion of the Coulomb branch moduli space is seen as coming from the field strength of the self-dual 2-form field of the $(0, 2)$ theory on the world volume of the 5-brane. In section four, we review mirror symmetry in three dimensions and then consider compactification of this symmetry to two dimensions. In the process we review Buscher's duality for the wormhole metric and the Taub-NUT metric and relate it to transformations in the brane construction. In section 6, we examine the monopole moduli space of five dimensional gauge theories and find two dual interpretations. In section 7, we consider two dimensional gauge theories with an adjoint hypermultiplet in addition to fundamental hypermultiplets, and we relate the two dimensional mirror symmetry to the duality between the $(1, 1)$ and $(0, 2)$ string theories. In section 8, we propose that there is a Seiberg-type duality in two dimensions that relates different Higgs branches to the same conformal field theory. We end by speculating about a quantum Higgs branch.

2. Field Theory on the D2 brane

2.1. The Brane Construction

The configurations we will study involve three kinds of branes in type IIA string theory: a Neveu-Schwarz (NS) fivebrane, Dirichlet (D) fourbrane and Dirichlet twobrane. Specifically, the branes are:

- (1) NS fivebrane with worldvolume $(x^0, x^1, x^2, x^3, x^4, x^5)$ lives at a point in the (x^6, x^7, x^8, x^9) directions. The NS fivebrane preserves supercharges of the form¹ $\epsilon_L Q_L + \epsilon_R Q_R$, with

$$\begin{aligned}\epsilon_L &= \Gamma^0 \cdots \Gamma^5 \epsilon_L \\ \epsilon_R &= \Gamma^0 \cdots \Gamma^5 \epsilon_R.\end{aligned}\tag{2.1}$$

- (2) D fourbrane with worldvolume $(x^0, x^1, x^7, x^8, x^9)$ lives at a point in the $(x^2, x^3, x^4, x^5, x^6)$ directions. The D fourbrane preserves supercharges satisfying

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^7 \Gamma^8 \Gamma^9 \epsilon_R.\tag{2.2}$$

¹ Q_L, Q_R are the left and right moving supercharges of type IIA string theory in ten dimensions. They are (anti-) chiral: $\epsilon_R = -\Gamma^0 \cdots \Gamma^9 \epsilon_R, \epsilon_L = \Gamma^0 \cdots \Gamma^9 \epsilon_L$.

- (3) D twobrane with worldvolume (x^0, x^1, x^6) lives at a point in the $(x^2, x^3, x^4, x^5, x^7, x^8, x^9)$ directions. The D twobrane preserves supercharges satisfying

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^6 \epsilon_R. \quad (2.3)$$

It is easy to check that there are eight real supercharges satisfying equations (2.1)-(2.3), $\frac{1}{4}$ of the original supersymmetry of type IIA string theory. Each relation (2.1)-(2.3) by itself would break $\frac{1}{2}$ of the supersymmetry. Equations (2.1) and (2.2) are independent and together break to $\frac{1}{4}$. Equation (2.3) is not independent of (2.1) and (2.2) and hence breaks no more of the supersymmetry. Altogether the branes preserve $\frac{1}{4}$ of the supercharges. If one T-dualizes along the x^2 direction, one recovers the IIB construction of [1].

2.2. The Fields

The global R-symmetries of the $N = (4, 4)$ supersymmetric theory on the D2-brane arise from the Lorentz group of the ten dimensional space-time. We have a $Spin(4)$ symmetry in the directions x^2, x^3, x^4, x^5 and an $SU(2)_R$ symmetry of the coordinates x^7, x^8, x^9 . There is also the $SO(1, 1)$ Lorentz symmetry of the coordinates x^0, x^1 which is space-time for the two dimensional theories which we will be interested. In all we have $SO(1, 1) \times Spin(4) \times SU(2)_R$. These are the correct symmetries for the $N = (4, 4)$ two dimensional theory [11]. The ten dimensional $\mathbf{16}^+$ and $\mathbf{16}^-$ supercharges of the IIA string theory leave unbroken $(\mathbf{1}, \mathbf{2}, \mathbf{2})^+ \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})^-$ under the $SO(1, 1) \times Spin(4) \times SU(2)_R$.

Hypermultiplets in the fundamental representation of the gauge group arise from strings stretched between the D2-branes and the D4 branes. Scalars in these hypermultiplets transform as $(\mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2})$ under $SO(1, 1) \times Spin(4) \times SU(2)_R$. We can understand this as follows: in the brane construction Higgsing corresponds to breaking a D2-brane between two D4 branes and moving it in the x^7, x^8, x^9 direction. We also must include the component A_6 from the gauge field, which is the only surviving part of the gauge field permitted by these boundary conditions. Once we embed the IIA theory in M-theory, the x^{10} replaces A_6 as the forth scalar. We see the scalars of the Higgs branch naturally fall into representations $\mathbf{3} \oplus \mathbf{1}$ under the $SU(2)_R$ corresponding to the three complex structures of the hyper-Kahler Higgs branch.

In terms of $N = (2, 2)$ superfields, the Coulomb branch consists of twisted (vector) multiplets, Λ , and normal chiral multiplets, Φ . The $U(1)$ gauge field on the D2-brane lives in the twisted multiplet, and motion in the directions x^2, x^3, x^4, x^5 corresponds to the

scalar fields components of both Φ and Λ . These scalars are therefore charged under the $(\mathbf{1}, \mathbf{4}, \mathbf{1})$ of the $SO(1, 1) \times Spin(4) \times SU(2)_R$. The Coulomb branch, parameterized by the scalars, is characterized by a Kahler potential which satisfies a four-dimensional Laplacian

$$\partial_\Lambda \partial_{\bar\Lambda} K + \partial_\Phi \partial_{\bar\Phi} K = 0. \quad (2.4)$$

This determines the metric

$$ds^2 = \partial_\Phi \partial_{\bar\Phi} K d\Phi d\bar\Phi - \partial_\Lambda \partial_{\bar\Lambda} K d\Lambda d\bar\Lambda \quad (2.5)$$

and the antisymmetric tensor field

$$B = \frac{1}{4}(\partial_\Phi \partial_{\bar\Lambda} K d\Phi \wedge d\bar\Lambda + \partial_\Lambda \partial_{\bar\Phi} K d\bar\Phi \wedge d\Lambda). \quad (2.6)$$

The torsion is determined from $H = dB$.

It has been argued that the Coulomb branch and the Higgs branch flow to different and distinct theories in the infra-red [14]. The basis for this claim is that the superconformal theory which these gauge theories flow is known to have an $SU(2) \times SU(2)$ global symmetry. Since scalars must be singlets under this symmetry, the Higgs branch must flow to a theory with global symmetry coming from the $Spin(4)$ since it has scalars charged as $(\mathbf{1}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})$ under $SO(1, 1) \times Spin(4) \times SU(2)_R$. Likewise, the Coulomb branch must flow to a theory with global symmetry coming from the $SU(2)_R$ since it has scalars charged as $(\mathbf{1}, \mathbf{4}, \mathbf{1})$ under $SO(1, 1) \times Spin(4) \times SU(2)_R$. One must keep in mind that there is no moduli space in two dimensions and we will always be working in the Born-Oppenheimer approximation.

2.3. The Parameters

Giving a mass to a hypermultiplet in the brane picture corresponds to moving a D4 brane away from a D2-brane in the x^2, x^3, x^4, x^5 direction. Therefore, masses transform in the $(\mathbf{1}, \mathbf{4}, \mathbf{1})$ of the $SO(1, 1) \times Spin(4) \times SU(2)_R$ symmetries. The Fayet-Iliopoulos parameters, $\vec{\zeta}$, correspond to motion of the NS 5-branes in the x^7, x^8, x^9 direction, and therefore they transform in the $(\mathbf{1}, \mathbf{1}, \mathbf{3})$ of the $SO(1, 1) \times Spin(4) \times SU(2)_R$ symmetries. They are background hypermultiplets. The θ angle corresponds to the 5-th scalar of the $(0, 2)$ tensor multiplet on the worldvolume theory of the NS 5-brane. As will be discussed below, going to M-theory makes the θ angle manifest. The FI parameters can

be viewed as the Kahler deformations of the metric of the associated hyper-Kahler non-linear sigma model [15]. Adding the theta angle gives a “quaternionic Kahler form” $\vec{\zeta} + i\theta$, analogous to the complexified Kahler form used in $N = (2, 2)$ theories. The θ parameter is to be associated with deformations of the antisymmetric tensor 2-form field of the non-linear sigma model. The distance between the NS 5-branes in the x^6 direction is inversely proportional to the gauge coupling constant

$$\frac{1}{g_2^2} = \frac{x^6}{M_s g_s} \quad (2.7)$$

where M_s is the string scale.

3. Quantum Effects on the Brane

3.1. Hanany-Witten Transition

We will consider a $U(N_c)$ gauge theory with N_f fundamental hypermultiplets. The corresponding brane configuration has N_c D2-branes suspended between two NS 5-branes separated in the x^6 direction with N_f D4-branes intersecting the D2-branes at points in x^6 . One can move the D4 branes in the x^6 direction outside of the NS 5-branes. In doing so, N_f D2-branes are created. Strings between the D2-branes inside the NS 5-branes and the newly created D2-branes outside give rise to the hypermultiplets. Situations with different x^6 positions for the D4 branes have been found to be equivalent in higher dimensions [1]. Since the two dimensional case is related to higher dimensional cases by T-duality, we will assume different x^6 positions are equivalent here (see Figure 1).

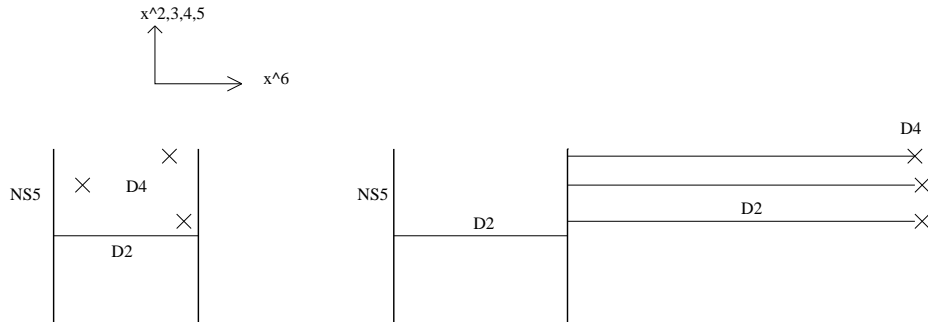


Fig. 1: The Hanany-Witten transition. The horizontal direction is x^6 . Both diagrams are equivalent from field theory.

Once the N_f D4 branes are outside of the NS 5-branes, we can consider what happens when the N_f D2-branes start to pull on the 5-branes. The tension of the D2-branes inside the NS 5-branes is negligible in comparison to the D2-branes on the outside since they are much shorter, and therefore we will not consider their effect on the deformation of the 5-brane. Since the tension of the NS 5-brane goes as $\frac{1}{g_s^2}$, bending of the branes is quantum mechanical in string theory. This can be seen by taking $g_s \rightarrow 0$. In this limit, the tension of the branes goes to infinity and there can be no bending. This is the classical limit for the world volume theory on the D-branes (which are non-perturbative objects in string theory). Quantum information about the field theory can therefore be gained from considering branes that bend (see Figure 2).

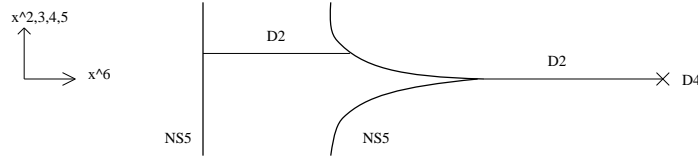


Fig. 2: A D2-brane pulls on the D5-brane, bending it as shown. The D2-brane doing the pulling becomes tensionless since there is nothing to balance it's tendency to collapse.

The physical situation of a D2-brane pulling on a NS 5-brane satisfies the four dimensional Laplacian, $\nabla^2 x^6 = \delta^{(4)}$, where the four dimensions are the ones on the 5-brane transverse to the 2-brane. A solution of the Laplacian is $x^6 = a/r^2 + c$ where $r^2 = (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2$. This solution minimizes the surface area of the 5-branes. We have for gauge group $U(1)$,

$$ds^2 = \left(\frac{1}{g_2^2} + \frac{N_f}{r^2} \right) (dr^2 + r^2 d\Omega_3^2) \quad (3.1)$$

where r is now a coordinate on the moduli space.² In field theory the modification to the flat metric comes from a one-loop calculation. There is also the antisymmetric tensor field

$$B = -\frac{1}{4} N_f \sin^2 \frac{\theta}{2} d\phi \wedge d\chi \quad (3.2)$$

² The coordinates on the moduli space and the coordinates in IIA space-time are related by M_s^2 , the string scale.

where $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, $0 \leq \chi < 4\pi$ are angular coordinates on the unit three sphere $S^3 \subset R^4$

$$\Phi = e^{i(\chi-\phi)/2} \cos \frac{\theta}{2}, \quad \Lambda = e^{i(\chi+\phi)/2} \sin \frac{\theta}{2}. \quad (3.3)$$

H is non-zero when the B -field is not well defined on S^3 giving the moduli space torsion. We will see below that (3.2) can naturally be associated with the $B_{\mu\nu}$ field of the worldvolume theory on the NS 5-branes of IIA. What we see is that as the 2-brane moves on the 5-brane it encounters singularities at the points where there is another 2-brane pulling on the 5-brane from the other side. This is the analogue of the *log* singularity found in the 4d case in [8]. The differences here are that the $1/r^2$ singularity is not corrected by instantons and that the singularity is infinitely far away from any point on the Coulomb branch. This metric (3.1) is identical to the wormhole solution of the NS 5-brane found in [10]. From the perspective of [10], N_f is the amount of charge coupling to the field strength $H = dB$, the torsion.

What is intriguing about the situation here is that the D4 branes can actually be at finite distance from the NS 5-branes. In this configuration it is possible to “go down” the throat of the wormhole and see what is at the bottom. We find that there is a tensionless D2-brane where the Coulomb branch meets the Higgs branch. This is consistent with there being a non-trivial fixed point at the origin of the Coulomb and Higgs branches; when the world volume of the brane probe comes into contact with a brane of zero tension, there is often a non-trivial fixed point [6]. Presumably, this is related to the new massless states that arise when a brane becomes tensionless and the fact that the theory becomes scale invariant as $g_{cl} \rightarrow \infty$.

It was shown [10] that the non-linear sigma model having the 5-brane as a target space is a level N_f supersymmetric $SU(2)$ WZW model with a Liouville field associated with the dilaton. Near the singularity, the metric (3.1) can be rewritten as

$$ds^2 = \frac{N_f}{r^2} (dr^2 + r^2 d\Omega_3^2). \quad (3.4)$$

We can introduce a new radial coordinate $\eta = \sqrt{N_f} \log(r/\sqrt{N_f})$ to give:

$$ds^2 = (d\eta^2 + N_f d\Omega_3^2), \quad (3.5)$$

The field η blows up as we proceed down the throat of the wormhole. Consequently, the Liouville field and the dilaton also blows up. This is therefore a singular conformal field theory. The topology here is $\mathbf{R}^+ \times \mathbf{S}^3$. In the brane picture, as the D2-brane extends

down the wormhole, the theory essentially becomes three dimensional. This is consistent with the idea [16] that the extra dimension of the D2-brane, x^6 , can be associated with a Liouville field. Since we find new massless states at the bottom of the throat, we conclude that there should perhaps be another description that replaces the CFT of [10], in the limit that the dilaton blows up.

3.2. Torsion

We interpret the torsion of the moduli space of the Coulomb branch in the brane picture in the following way. When N_f D2-branes end on an NS 5-brane of type IIA string theory, the D2-branes look like strings in a 5+1 dimensional theory. Strings in six dimensions couple to the self-dual 2-form field, $B_{\mu\nu}$. The charge that the strings carry is therefore $N_f = \int_{S^3} H$ where $H^{(3)} = dB$. In two dimensional sigma models, torsion comes from a $B_{\mu\nu}$ field and is equal to $H^{(3)}$ [17]. Each fundamental hypermultiplet, contributes an integer amount to the torsion of the Coulomb branch. We therefore identify (2.6) with the 2-form field of the six-dimensional $(0, 2)$ theory.

A D2-brane ending on a D4-brane also looks like a string, but in 4+1 dimensions. The string couples electrically to the $B_{\mu\nu}$ field, which is the dual of the photon on the worldvolume theory on D4 brane. The magnetic charge carried by each string is therefore $N_f = \int_{S^2} F^{(2)}$ where $F = dA$. Each D2-brane looks like a monopole in the D4-brane. Monopole moduli space is given by a Taub-NUT metric which is torsion free. This is consistent with there being no torsion on the Higgs branch since motion of a D2-brane between two D4-branes corresponds to Higgs branch moduli. We will see below that in M-theory the metric on the Higgs branch is better thought of as being T-dual to the Taub-NUT.

This analysis is also consistent with there being no torsion for the Coulomb branch in three dimensions. The three dimensional Coulomb branch is described by a D3-brane ending on an NS 5-brane in IIB string theory. The D3-brane looks like a 2-brane in 5+1 dimensions which couples electrically to a 3-form field $C^{(3)} = *A^{(1)}$, the dual of the photon in six dimensions. The D3 brane is therefore a monopole on the NS 5-brane. There is no coupling to a $B_{\mu\nu}$ field, $H = 0$, and the Coulomb moduli space has no torsion. Likewise, a D4-brane ending on an NS 5-brane of IIA describes Coulomb branch of the four dimensional theory. There is no torsion here either since the D4-brane, being a vortex solution in the 5+1 theory, does not couple to the 2-form field. Again this is consistent with there being no torsion on the Coulomb branch moduli space of theories in four space-time dimensions.

3.3. The view from M-theory

In this section, we look at the brane configuration in M-theory and attempt to understand the infra-red dynamics of the associated gauge theory. Let's start in IIA where the compact direction x^{10} is very small. As explained in section 2.3, motion of the NS 5-branes in the x^7, x^8, x^9 direction corresponds to an FI term $\vec{\zeta}$. Once the direction x^{10} becomes of finite radius, there is a new direction in which the 5-brane can move. This is the theta angle, θ . Notice that a non-zero theta angle breaks supersymmetry if all the hypermultiplets have zero vacuum expectation value $\langle Q \rangle = 0$. This is consistent with the interpretation of the theta angle as an electric field since it creates a non-zero vacuum energy.³ As was discussed in [19], the point where $\langle Q \rangle = \langle \Phi \rangle = \vec{\zeta} = \theta = 0$ is the point where the Higgs branch meets the Coulomb branch. Therefore, the wavefunction on the Higgs branch can move onto the Coulomb branch. In [19], this was interpreted as evidence for the conformal field theory on the Higgs branch becoming singular.

What is interesting, now that the eleventh dimension has become big, is that we can see that the

$$SO(1, 1) \times Spin(4) \times SU(2)_R$$

global symmetries have become enhanced to

$$SO(1, 1) \times Spin(4)_H \times Spin(4)_C.$$

In [14] it was argued that the theory on the Coulomb branch decouples from the theory on the Higgs branch and flows to a superconformal fixed point with an $SU(2) \times SU(2)$ symmetry. Since the scalars of the Coulomb branch transform under the $Spin(4)_H$, this is not a candidate. It was conjectured in [14] that the $SU(2)_R$ gets enhanced to $Spin(4)_C$. Here we see a realization of that idea; the opening up of the eleventh dimension makes the Lorentz group larger which appears in the brane construction as an enhanced R-symmetry. What's more, we see that there is a natural symmetry between the $Spin(4)_H$ and the $Spin(4)_C$ exchanging the Coulomb and Higgs branches, and the mass parameters of the fundamental fields with the FI parameters and theta angle. This is much like the relationship between the R-symmetries in three dimensions between the $SU(2)_H$ of the Coulomb branch and the $SU(2)_C$ of the Higgs branch. Notice, however, there is an important difference. There is no S-duality in M-theory, as there was in IIB theory, that

³ This was independently noted in [18].

enables us to transform the Higgs branch into the Coulomb branch as was done in [1]. There is however another IIA theory with a “mirror” Higgs branch and Coulomb branch that becomes equivalent to this theory in eleven-dimensional M-theory.

The two IIA brane configurations that flow to the same configuration in M-theory are in fact the same “mirror” pairs that were discussed in three dimensions in [4]. For example, a $U(1)$ gauge theory with N_f flavors is “mirror” to a $U(1)^{N_f-1}$ gauge theory with matter charged in the $(1, 0, 0, 0, 0, \dots, 0)$, $(1, -1, 0, 0, 0, \dots, 0)$, $(-1, 1, 0, 0, 0, \dots, 0)$, $(0, 1, -1, 0, 0, \dots, 0)$, $(0, -1, 1, 0, 0, \dots, 0)$, ..., $(0, 0, 0, 0, 0, \dots, 0, 1)$. The Higgs branch of this mirror theory is classically an A_{N_f-1} ALE space. However, unlike usual orbifold theories here we have $\theta = 0$ since we consider no separation between the 5-branes in the x^{10} direction. This is a singular limit in the conformal field theory describing the ALE space [20,19]. As discussed above, the $U(1)$ theory with N_f flavors is constructed by suspending a D2-brane between two NS 5-branes with N_f D4-branes intersecting the D2-brane at points in the x^6 direction. The mirror theory is constructed by suspending a D2-brane between two D4-branes with N_f NS 5-branes intersecting the D2-brane at points in the x^6 direction. Upon going to M-theory, the D4-branes become 5-branes, the NS 5-branes remain 5-branes, and the D2-branes remain membranes. Therefore, the two constructions which were different in IIA string theory are equivalent in M-theory.

4. Compactification of the theory from three dimensions to two dimensions.

4.1. Review of 3d mirror symmetry

Here we review the mirror symmetry of [4], and in the next section explore it’s reduction to two dimensions. The three dimensional construction of the branes was carried out in [1]. It is equivalent to our construction in section 2.2 once we T-dualize upon x^2 . We start in IIB string theory with N_c D3 branes in directions (x^0, x^1, x^2, x^6) stretched between NS 5-branes in directions $(x^0, x^1, x^2, x^3, x^4, x^5)$. Motion of the D3 branes in the x^3, x^4, x^5 direction constitutes motion on the Coulomb branch. The Higgs branch is provided by N_f D5 branes in $(x^0, x^1, x^2, x^7, x^8, x^9)$ which allows the D3 brane to break on the D5 branes and move in the x^7, x^8, x^9 direction. The fourth scalar of the quaternionic Higgs branch is provided by the A_6 component of the gauge field. Let’s consider, for definiteness, a $U(1)$ theory with $N_f = 1$ although the following statements will be fairly general. We have one D3 brane suspended between two NS 5-branes and one D5 brane intersecting the D3-brane at a point in x^6 . We consider performing the Hanany-Witten transition and putting the D5

brane outside of the NS 5-branes, creating a new D3 brane stretched between the NS brane and the D5-brane. The D3 branes pull on the NS 5-branes creating a $1/r$ singularity. As before, we interpret this as meaning that the Coulomb branch has an $1/r$ singularity. If we dualize the photon we get a compact scalar. In the IIB string theory, dualizing the photon corresponds to performing an S-duality which turns the NS 5-branes into D5-branes and takes strong to weak coupling. The dual photon is the A_6 component of the D3 brane which is not projected out by the D5 brane boundary conditions, while the fields A_1 and A_2 are projected out. In the dual variables the Coulomb branch is $\mathbf{R}^3 \times \mathbf{S}^1$, the S^1 coming from the compact scalar A_6 . Quantum mechanically, the Coulomb branch is modified to a Taub-NUT metric.

$$ds^2 = g_3^2(\vec{x})(d\sigma + \vec{\omega} \cdot d\vec{x})^2 + g_3^{-2}(\vec{x})d\vec{x} \cdot d\vec{x}, \quad (4.1)$$

with

$$g_3^{-2}(\vec{x}) = g_{cl}^{-2} + \sum_{i=1}^{N_f} \frac{1}{|\vec{x} - \vec{m}_i|}, \quad \vec{\nabla}(g_3^{-2}) = \vec{\nabla} \times \vec{\omega}. \quad (4.2)$$

where $\vec{x} = (x^3, x^4, x^5)$ and σ is the compact direction A_6 . For $N_f = 1$ there is a removable singularity in the metric. In the brane configuration, we now have D3-branes pulling on a D5-brane. The singularity due to the bending is just (4.2) where g_{cl} is the separation between the D5 branes in x^6 far from the singularity. As we approach the singularity, the effective coupling of the three dimensional gauge theory goes to zero since x_6 goes to infinity. Since the radius of the S^1 fiber of the Taub-NUT metric (4.1) is proportional to g_3 , the fiber shrinks to zero radius at $r = 0$. In this limit, we can neglect the constant c , and the metric goes over to the ALE metric (see Figure 3).

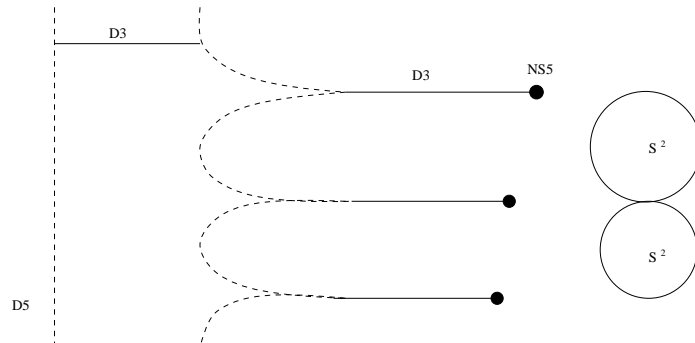


Fig. 3: We see how the bending of the D5-branes creates a Taub-NUT metric. S^2 indicates that there is a 2-cycle whose radius depends on the distance between the NS 5-branes (the FI terms).

We saw above, that the singularity at the origin of the Coulomb branch has a description in terms of dual (magnetic) coordinates as a Taub-NUT metric. As g_{cl} goes to infinity, the singularity is also described by an ALE singularity of a classical mirror Higgs branch, which from the perspective of the branes is the same as the dual coordinates of the Coulomb branch. This is seen as evidence [4] for a non-trivial fixed point at the origin of the Higgs and Coulomb branches for $N_f > 1$ for a $U(1)$ gauge theory. At $r = 0$, it appears that there is a tensionless D3 brane for reasons similar to those argued in section 3.1. This would be consistent with there being a non-trivial fixed point at the origin of the Higgs and Coulomb branches [6].

4.2. Buscher duality.

It is well known that if one T-dualizes along the S^1 of the Taub-NUT space one gets a wormhole metric with one of it's transverse directions compactified (also called an H-monopole) [21,22]. Let's review this duality. In the metric (4.1), the direction σ is compact. This is an isometry; we can dualize upon this. Rewriting (4.1)

$$ds^2 = g_3^{-2}(\vec{x})d\sigma d\sigma + g_3^2(\vec{x})\omega_i d\sigma dx^i + g_3^2(\vec{x})\omega_i dx^i d\sigma + (g_3^2(\vec{x})\omega_i \omega_j + g_3^{-2}(\vec{x})\delta_{ij})dx^i dx^j \quad (4.3)$$

Gauging the isometry and then integrating out the gauge fields as described in [23] and [24], we find that

$$\begin{aligned} ds^2 &= g_3^2(\vec{x})(d\sigma d\sigma + \delta_{ij}dx^i dx^j) \\ b_{\sigma i} &= \omega_i \\ \Phi &= \log(g_3^{-2}(\vec{x})) \\ &= \log(g_{cl}^{-2} + \sum_{i=1}^{N_f} \frac{1}{|\vec{x} - \vec{m}_i|}) \end{aligned} \quad (4.4)$$

The \log appears because under T-duality the ratio $\frac{e^{2\Phi}}{R}$ must be preserved. We see that the cross-terms in (4.3) have produced the antisymmetric tensor field of (4.4). Moreover, we have produced the logarithmic dilaton and a metric that has the same form as the wormhole solution described in (3.1)[10]. However, there is a difference between this metric (4.4) and the one in (3.1); the $g_3^{-2}(\vec{x})$ given in (4.1) goes as $1/|\vec{x}|$ whereas the wormhole metric of [10] goes as $1/r^2$. The difference is clearly due to the fact that (4.4) the direction σ , which is transverse to the 5-brane, is compact, and therefore the appropriate Laplacian is three dimensional rather than four dimensional. Moreover, we are neglecting states that

propagate in the compact direction. When we decompactify the S^1 , the $g_3^{-2}(\vec{x})$ of (4.4) becomes $1/|\vec{x}|^2 + |\sigma|^2$, we must include states that propagate in the compact direction, and, as noted in [11], the wormhole metric no longer has an isometry and therefore we do not know how to dualize. Momentum modes that probe the wormhole metric are dual to strings that wind around the S^1 of the Taub-NUT. In [21], it was found that the winding modes should see a more “throat-like” behavior of the Taub-NUT.

4.3. Dimensional reduction from 3 to 2

In [11], it was conjectured that the wormhole metric on the Coulomb branch has another interpretation in terms of some unspecified dual coordinates, where the metric is ALE. We will here attempt to understand this speculation in terms of brane constructions. Consider compactifying the direction x^2 of the IIA set up in section 2.2 and wrapping the NS 5-brane on an S^1 of small radius R_2 . The bending of the branes gives us the metric (4.4) with a $1/r$ singularity. By T-duality, we can turn this into a large circle of radius $R_B = 1/M_s^2 R_2$ which also takes the D2-brane to a D3 brane, and a IIA 5-brane with a B field to a IIB 5-brane without a B-field. This is the set-up of 4.1. Although from the point of view of the NS 5-brane the D3 brane is a monopole and the moduli space of monopoles should have a Taub-NUT metric, in field theory the Taub-NUT is only visible once we dualize the compact scalar. In the brane theory that corresponds to an S-duality of IIB.

In the limit that R_2 becomes big, the description of the metric in the IIA theory is better described by (3.4) which has a $1/r^2$ singularity. T-dualizing this to IIB, we have a D3 brane wrapped on a small circle with radius R_B . This is a 3d theory on a $R^{1,1} \times S^1$ base space where the scalar parameterizing the Wilson loop is big. If we now dualize the photon, which corresponds to performing S-duality on the IIB configuration, this takes us to a configuration of D5 branes and D3 branes wrapped on x^2 . From the perspective of the D5 brane, the D3 branes are monopoles. Therefore, the metric on the moduli space of D3 branes is given by a Taub-NUT metric (4.1) which in the limit that g_{cl} goes to infinity becomes the ALE metric. This appears to be a realization of the speculation in [11] that there are some coordinates in which the wormhole metric is ALE. However, once we T-dualize on x^2 , taking the S-dual IIB configuration to IIA, we find a configuration of D2-branes suspended between D4 branes. The metric here is also ALE with $\theta = 0$. This sigma model has no well defined conformal field theory. As we open up the eleventh dimension, the fourth scalar of the Higgs branch no longer comes from A_6 but rather from x^{10} . The metric is now better thought of as the metric (4.4) rather than ALE. As the radius

of the eleventh dimension becomes large, the D4 brane decompactifies into a 5-brane. The fact that we can give non-zero values to the fourth scalar, x^{10} , and parameter, θ , implies that there should be four parameters, $\vec{\zeta} + i\theta$, to tune such that there is a singularity, unlike (4.1) which only has three parameters to tune. It is clear that this must be the case, since as the 11-th dimension opens up, we see that the Laplacian of the D2-brane pulling on the D4-brane, goes from being a three dimensional Laplacian to being the four-dimensional Laplacian of a D2-brane pulling on a 5-brane. The metric on the Higgs branch is therefore (3.1). We said before that deformations of the antisymmetric 2-form of the non-linear sigma model are to be associated with θ , the distance between the 5-branes in the eleventh dimension. Allowing the eleventh dimension to open up, allows for a new deformation to the corresponding non-linear sigma model, the antisymmetric tensor field. H is non-zero at the point where B is not well defined. This point where H is non-zero occurs in the branes when θ becomes non-compact. This is consistent with the appearance of torsion in the metric (3.1).

Since the Higgs branch of the mirror theory is ALE, it is hyperKahler and such sigma models do not receive quantum corrections. On the other hand, since we are considering the point where there is no well defined CFT, $\theta = 0$, it is not clear that the non-renormalization theorems apply. Moreover, in [19], it was conjectured that the ALE orbifold theory with $\theta = 0$ flows to the CFT of [10], discussed in section 3.1. Here we see a realization of that idea and a resolution of a puzzle: In [19], it was not clear how the ALE metric would develop torsion. Here we see that torsion comes about when the gauge field of the 4+1 theory becomes the self-dual 2-form of the (0,2) 5+1 dimensional theory at strong coupling. Since the D2 branes in the 5-brane look like strings charged under H , the moduli space develops non-zero torsion. Branes allow us to see such novel non-perturbative effects explicitly.

5. Monopoles in 4+1 SYM

If the theory of D2-branes suspended between D4 branes is, from the D4 brane perspective, a monopole moduli space, what happens when we go to M-theory? What happens to the monopoles? The answer is that the 4+1 Super Yang-Mills theory is not well defined; it flows to a free theory in the IR. In order to have a well defined theory, we must embed the 4+1 SYM into the (0,2) 5+1 theory that was described in [25][26]. If we want quantum information about the 4+1 SYM, we must consider the 5+1 (0,2) theory. Magnetically

charged objects in the 4+1 SYM theory are strings. When we go to the (0,2) theory, the strings remain strings. However, from the point of view of the 5+1 theory, the strings are co-dimension four objects. The string moduli space of the 5+1 (0,2) theory is the moduli space of monopoles of the 4+1 SYM. Therefore, the metric on the moduli space of monopoles should be something that interpolates between $1/r$ and $1/r^2$. We know of such a metric. It is (4.4). This is perhaps to be expected since the gauge field in 4+1 comes from the $B_{\mu\nu}$ field of 5+1 as is the case for H-monopole solutions [22]. Hence, there are two descriptions of the monopole moduli space in five dimensions: the IIA perspective where the metric is Taub-NUT and the M-theory perspective where the metric is H-monopole (4.4).

6. Theories with an adjoint hypermultiplet

6.1. The metric

It is easy to generalize the discussion above to $U(N_c)$ gauge theories with N_f flavors and an adjoint hypermultiplet. Instead of having the N_c D2-branes end of NS 5-branes, we compactify the D2-branes on a circle in x^6 and dispense with the NS 5-branes. The metric that the D2-brane sees is the metric produced by the D4-branes in ten dimensional space-time, which generically goes like $1/r^3$. However, since we are taking dimension x^6 to be very small, the D4 brane metric is effectively $1/r^2$. This is the metric on the Coulomb branch of the two dimensional gauge theory (at least for the case of the $U(1)$ gauge field). The mirror theory with $U(1)^{N_f}$ with bifundamental matter fields is constructed by intersecting the D2-brane at points on the circle in the x^6 direction by NS 5-branes. If we consider the Higgs branch of the mirror theory, then the ten dimensional metric that D2-brane now sees is not ALE, but rather the T-dual of an ALE space (4.4) since the type IIA 5-branes with one direction compact are T-dual to Kaluza-Klein monopoles of IIB. Going to M-theory decompactifies the direction transverse to the 5-brane of IIA and takes the corresponding metric (4.4) to the wormhole metric (3.4).

6.2. Relations to the Matrix descriptions of the (1,1) and (0,2) string theories

As was explained in [25], the theories on the NS 5-branes of type IIA and IIB string theory decouple from the bulk in the limit that the string coupling constant g_s goes to zero while the tension of the strings M_s is held fixed. We will show here that the two dimensional mirror symmetry, discussed above, relates the two decoupled six dimensional string theories

to each other ⁴. A Kaluza-Klein monopole in directions $(x^0, x^1, x^2, x^3, x^4, x^5, x^6)$ in M-theory with the direction x^{10} compactified on a large circle R_{10} can be described by IIA string theory with a D6 brane in the limit $g_s^A \rightarrow \infty$ since $g_s^A = M_s R_{10}$. If the radius of the direction x^6 , $R_6 \ll 1/M_s$, then the theory is better described as a IIB theory with a D5 brane. The string coupling of the IIB theory is

$$g_s^B = \frac{R_{10}}{R_6} \rightarrow \infty. \quad (6.1)$$

S-duality takes the D5 brane to an NS 5-brane, and inverts the coupling constant. This is therefore the limit in which the IIB NS 5-brane decouples from the bulk. This is also the limit in which the mirror symmetry is valid.

On the other hand, the KK monopole of M-theory has a Matrix description in terms of D0 branes and KK monopoles of IIA[9]. We choose the direction x^5 to be the infinite boost direction. Therefore, here the IIA string coupling is $g_s^A = M_s R_5$ where $M_s^2 = M_{pl}^3 R_5$. The coupling constant of the world volume theory on the D0 brane is

$$g_1^2 = M_{pl}^6 R_5^3.$$

Since the radius R_6 is small, we can T-dualize this into a big circle. We now have a IIB theory with a D1 brane in directions (x^0, x^6) , parallel to a KK monopole of IIB. T-dualizing on x_{10} , the NUT direction, we have a theory of D2-branes in directions (x^0, x^6, x^{10}) on a small circle intersecting NS 5-branes in $(x^0, x^1, x^2, x^3, x^4, x^6)$ [9]. This theory has a Higgs branch metric that is ALE. The coupling constant on the 1+1 dimensional theory is

$$\frac{1}{g_2^2} = \frac{R_{10} R_6}{R_5} \tilde{R}_{10} = \frac{R_6}{M_s g_s^A}$$

where $\tilde{R}_{10} = \frac{1}{M_s^2 R_{10}}$. In the limit where $g_s^A \rightarrow \infty$, $g_2^2 \rightarrow \infty$. This is also the limit where (6.1) the physics of the IIB 5-brane decouples from the bulk.

The relationship between the 1+1 $U(N_c)$ gauge theory with N_f hypermultiplets and an adjoint hypermultiplet is more direct. Consider the D2-branes in x^0, x^1, x^6 and the D4 branes in x^0, x^1, x^7, x^8, x^9 . T-dualizing along x^6 takes us to a IIB configuration with D1 branes and D5 branes. This is the Matrix string configuration description of the IIA 5-brane discussed in [14]. The limit in which the string coupling constant goes to zero corresponds to the strong coupling limit on the D2-brane. We see then that the two dimensional mirror symmetry between $U(N_c)$ with N_f fundamentals and an adjoint hypermultiplet and $U(N_c)^{N_f}$ with bifundamental matter is a relation between the string theories on the NS 5-branes of IIA and IIB respectively in the limit that they decouple from the bulk [9].

⁴ We thank A. Hanany for suggesting this.

7. Seiberg Duality in Two Dimensions

In four dimensions with 4 real supercharges, there is a duality between an $SU(N_c)$ gauge theory with N_f flavors and an $SU(N_f - N_c)$ gauge theory with N_f flavors. There are no adjoint matter fields in these theories, and the dimensions of both Higgs branch moduli spaces are the same. This duality was realized in terms of brane constructions in [1,27]. Generically for theories with 8 supercharges the duality is spoiled by the adjoint matter fields. Seiberg duality in two dimensions could exist since the theory on the Higgs branch decouples from the theory on the Coulomb branch. One can check that \hat{c} is the same for the $SU(N_c)$ gauge theory with N_f flavors and the $SU(N_f - N_c)$ gauge theory with N_f flavors on the Higgs branch. Furthermore, from the brane perspective, we can perform the same operations that were carried out in [27] T-dualized from four to two dimensions. It is natural to expect that there are such dual conformal field theories on the Higgs branch.

8. The quantum Higgs branch

Classically a two-dimensional $U(N_c)$ gauge theory with $N_f = 1$ flavors and eight supercharges has no Higgs branch. In [14], there was speculation that quantum mechanically there could be such a Higgs branch. Motivation for this speculation came from Matrix theory where the theory of one $(0, 2)$ NS 5-brane is conjectured to be non-trivial. We have seen here that the metric describing the dual Higgs branch should, according to the brane picture, become deformed, due to non-trivial IR dynamics, into a something that has a Kahler potential that goes as $1/r^2$. Although we do not know exactly what this metric is, it seems reasonable to expect that the singularity in the metric for $N_f = 1$ will not be a coordinate singularity as it is for an ALE metric, but rather a real singularity. A real singularity would imply that there is a non-trivial Higgs branch for $N_f = 1$.

9. Conclusions

In this paper we studied the Coulomb and the Higgs branches of $N = (4, 4)$ theories in two dimensions. We found using a brane construction, two different two dimensional theories that become equivalent in the IR. In eleven dimensions there is a manifest symmetry that exchanges the Higgs branch of one theory with the Coulomb branch of another theory. The branes provide a way of seeing the $SU(2) \times SU(2)$ symmetry of the conformal

field theory to which the Coulomb branch flows. We have seen that ALE sigma models with $\theta = 0$ flow to wormhole CFTs of [10].

In [11] it was conjectured that the “wormhole” singularities of the Coulomb branch can be described in terms of some unspecified dual variables as ALE singularities of a dual Higgs branch. This idea was inspired by [28] where it was conjectured that ALE singularities in string theory are dual to 5-branes by mirror symmetry (or rather “fiber-wise T-duality”). Like the mirror symmetry between Calabi-Yau manifolds, such conjectures are difficult to prove stemming from the fact that there are often no isometries to dualize upon [13] (although it is possible sometimes to dualize upon broken isometries [29]). This is the case with the conjecture of [28] that ALE singularities are mirror to 5-branes. We have given here an interpretation from the brane perspective of this duality and explained some puzzles about the development of torsion on the ALE side. We suggest that brane configurations can be useful tools when analyzing sigma models with no well defined CFT.

It is interesting to note that the D-brane configuration that was considered in this paper, a D2-brane ending on an NS 5-brane, has been studied as a theory of self-dual strings in six dimensions. Little is known about self-dual strings since they do not allow for a perturbative description. Also, if one T-dualizes this set-up such that the 5-branes become ALE singularities, one recovers D3 branes of IIB wrapped on S^2 cycles of an ALE space another situation where self-dual strings arise.

Finally, considering all the progress that has been made in finding exact solutions of higher dimensional theories, one might ask why return to two-dimensions. In this paper we analyzed certain $(4, 4)$ theories via brane constructions. However, brane constructions of 2d, 3d, 4d, 5d, and 6d with 8 real supercharges are all related to each other by T-duality, hence correspondences between dimensions are manifest. A motivation for studying 2d theories is the hope that it will be possible also to travel up in dimension and relate exact solutions in 2d of interacting fixed points to higher dimensions where SCFTs are known to exist, but as of yet have not been solved. Although in 2d there is the power of the two dimensional superconformal algebra at one’s disposal, one can argue that the branes see no difference between 2d and other dimensions and so one should expect similars solutions. Understanding precisely the relations between the branes and the superconformal 2d fixed points is essential for such a program.

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